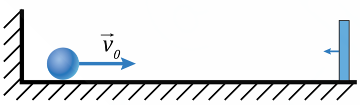
Adiabatic Invariants

- Answers and detailed solutions to all problems are provided in iOS/Android "**PhysOlymp**" app
- With any suggestions please write to **feedback@physolymp.com**

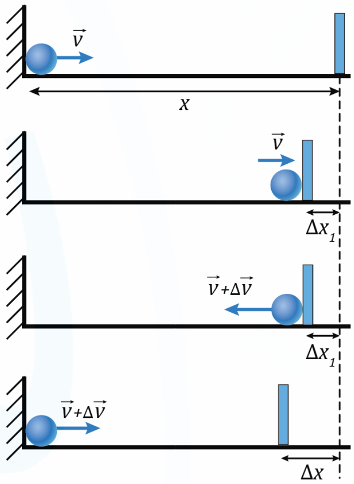
A special approach is required for systems with cyclic behavior and slowly changing parameters. If during one full period of time changes of the specific characteristics of the cyclic system are very small, it is usually possible to derive a correlation between variable parameters, which narrows down to some constant called "adiabatic invariant". General tips for solving this particular type of physics problem are not much different from any other type of problems, such as making several snapshots of the system at different periods of time, looking at the small changes of parameters with neglecting the smallest components in equations and finally combining several of them. More specific recommendation is thorough examination of the system during one full cycle at some arbitrary moment of time and look for small changes in length, momentum or something else and try to correlate those small changes with other changes occurred during that time. Also many problems can be solved with using energy conservation law, by examining average losses and gains in energy during one full cycle

Example 1

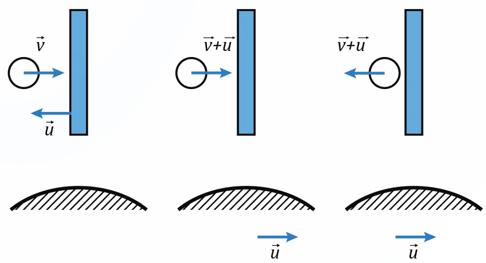
A small body is moving along flat frictionless horizontal surface, bouncing between two massive vertical walls. One of the walls is also moving but with a velocity, which is much less than of the ball at any time. Find velocity of the ball v_f at the moment when distance between the walls will reduce by 20%. Assume that collisions with the walls are perfectly elastic, initial velocity of the ball is v_0



Let's look at the system at some arbitrary moment of time, when velocity of the body is v and distance between two walls is x



During collisions with a moving obstacle, the velocity of the ball is slightly changing by Δv, while during impact with a stationary wall, it only changes direction, but not magnitude of the velocity. If the wall is moving with some constant velocity u, then in the reference frame of the moving wall, the ball approaches it with a velocity $(v+u)$ and bounces back in mirror like manner



Then velocity of the ball after collision with the moving wall in laboratory reference system will be

$$v+\Delta v = (v+u)+u$$

Or

$$\Delta v = 2u$$

During one full period of time T, distance between the walls reduces by

$$\Delta x = -uT$$

where negative sign indicates that distance is reducing with time
As $v \gg u$

$$T \approx \frac{2x}{v}$$

Combining the last three equations results in following relation between changing parameters

$$\Delta x = -\frac{\Delta v}{2} \cdot \frac{2x}{v}$$

Integrating both parts of the equation

$$\int \frac{dx}{x} = -\int \frac{dv}{v}$$

$$\ln x = -\ln v + const$$

Or

$$xv = const$$

Such kind of correlations with a slowly changing parameters is adiabatic invariant

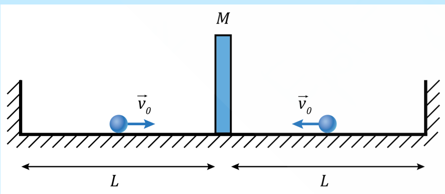
Substituting initial and final conditions of the system into correlation:

$$Lv_0 = 0.8Lv_f$$

$$\boxed{v_f = 1.25v_0}$$

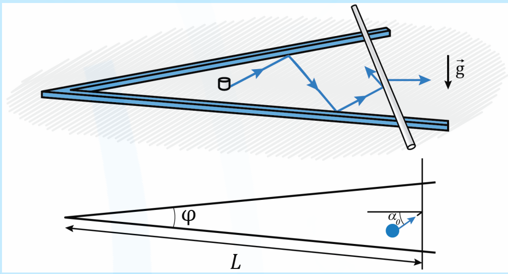
Problem 1

Two small balls are bouncing between vertical walls and heavy brick as shown at the picture below. The brick can move along flat horizontal surface without friction, all collisions of the balls are perfectly elastic. At initial condition the brick is at rest with a distance L to both of the walls, while balls are moving with velocities v_0. Find period of oscillations of the brick T, when it is moved sideways at a small distance from a symmetrical initial state. Assume that T is much larger than time between any two collisions of the balls. Mass of the brick M is much larger than mass of the balls m



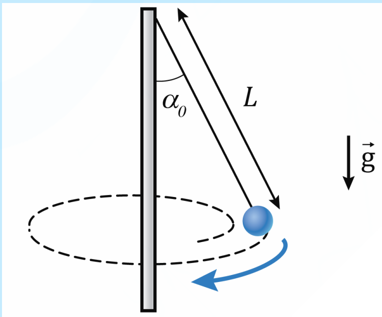
Problem 2

A small rigid disc is moving without friction along flat horizontal surface limited within a space in form of isosceles triangle with a small angle $\varphi \ll 1$. Base of the structure is moving very slowly with external force. All collisions of the puck with vertical walls of the triangle and its base are perfectly elastic, and all the walls are very heavy and smooth. At initial conditions the disc collides with the base of triangle at angle α_0. What will be incident angle α_f when dimensions of the triangle will increase in two times? Assume that puck is so small that it does not get stuck at the vertexes of the triangle and it is always in a stable vertical position (does not topple over)



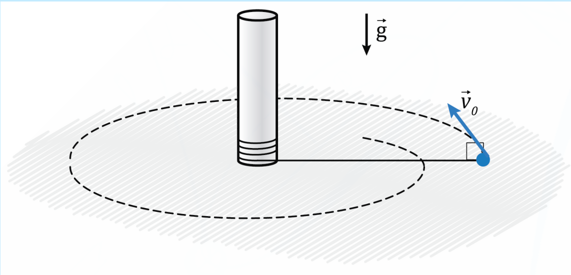
Problem 3

A small bob is attached to the non-stretchable weightless thread, which is winding around a very thin vertical pole. Initial length of the thread in the air is L_0, and initial angle with vertical $\alpha_0 = \pi/6$. Motion of the bob around the pole is close to circular with very slow changes in time for angle and length of the free thread. Assuming that thread is no sliding along the pole, find length L_f of the thread in the air when it has angle $\alpha_f = \pi/3$



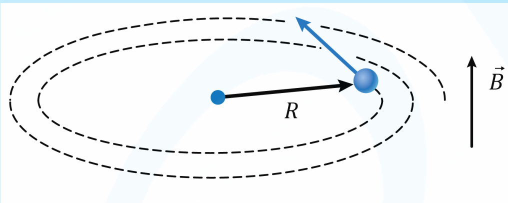
Problem 4

A small bead, which is attached to the weightless thin elastic band is moving without friction at the flat horizontal surface, wrapping around stationary cylinder. Assume that equilibrium length of the band is negligibly small and cylinder is sticky, so that after touching surface of the cylinder, elastic band is not slipping. Find velocity of the bead u when the length of the band is reduced by half from initial conditions, when velocity of the bead is v_0. Also assume that, length of the free part of the elastic band is much larger than radius of the cylinder and initial trajectory of the bead is close to the circular



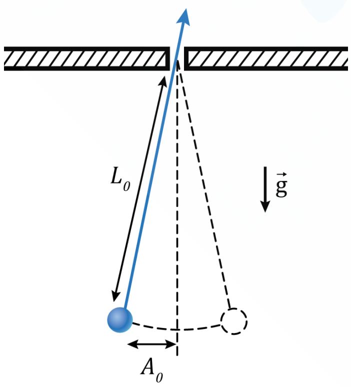
Problem 5

A small charged particle is moving in slowly changing magnetic field. Initial radius of the trajectory is r_0. Find radius of trajectory of the particle R when magnetic field will double



Example 2

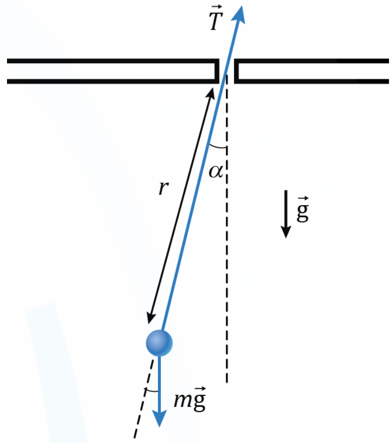
A simple pendulum is oscillating in gravity field, while its length is decreasing very slowly by pulling the rope through the hole in the ceiling. What will be amplitude of the small oscillations A_f when the length of the thread will be reduced by half? Initial amplitude of the bob is A_0



If length of the rope at some moment of time is r, then tension force of the thread as a function of deviation angle α should be

$$T = mg\cos\alpha + \frac{mv^2}{r}$$

' where m is mass of the bob and v - its velocity



As length of the rope is reducing very slowly, during any one full cycle, motion can be described as of a simple pendulum with frequency of oscillations

$$\omega^2 = \frac{g}{r}$$

Hence angle α and velocity of harmonic oscillations v can be described as

$$\alpha = \alpha_A \sin(\omega t + \varphi)$$

$$v = r\dot{\alpha} = r\alpha_A\omega\cos(\omega t + \varphi)$$

where α_A is maximum deviation angle during one cycle, φ is a phase shift, which is related to the reference point of the measured time t
For small angles

$$\alpha \ll 1 \quad \cos\alpha \approx 1 - \frac{\alpha^2}{2}$$

Substituting everything into equation with tensition force

$$T \approx mg\left(1 - \frac{\alpha^2}{2}\right) + m\dot{\alpha}^2 r$$

$$= mg\left(1 - \frac{\alpha_A^2\sin^2(\omega t + \varphi)}{2}\right) + m\alpha_A^2\omega^2 r\cos^2(\omega t + \varphi)$$

To pull the rope during one cycle, external force should do a work

$$\Delta W = -\langle T\rangle \Delta r$$

where $\langle T\rangle$ is average tension force during one cycle, Δr - is reduction in pendulum length during that time. Negative sign indicates that for reduction of length ($\Delta r < 0$) should be done a positive amount of work

$$\langle T\rangle = mg\Delta r - \frac{mg\alpha_A^2}{2}\left\langle \sin^2(\omega t+\varphi)\right\rangle \Delta r + m\alpha_A^2\omega^2 r\left\langle \cos^2(\omega t+\varphi)\right\rangle \quad (1)$$

Averages of trigonometric fuctions for one full period of time are

$$\sin^2(\omega t+\varphi) = \left\langle \frac{1-\cos 2(\omega t+\varphi)}{2}\right\rangle = \frac{1}{2} - 0 = \frac{1}{2}$$

$$\cos^2(\omega t+\varphi) = \left\langle \frac{1+\cos 2(\omega t+\varphi)}{2}\right\rangle = \frac{1}{2} + 0 = \frac{1}{2}$$

Combining last few equations, amount of work done by pulling the rope during one period is

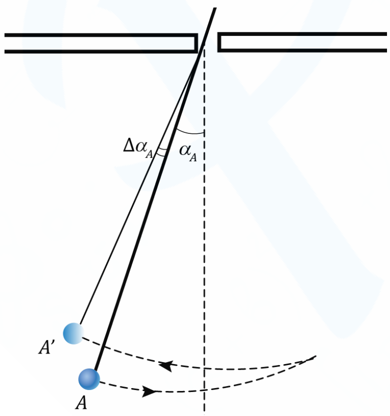
$$\Delta W = -mg\left(1+\frac{\alpha_A^2}{4}\right)\Delta r \quad (2)$$

That work is tranformed into increasing potential energy of the bob

$$\begin{aligned}\Delta E_p &= mg(r-\Delta r)\cos(\alpha_A+\Delta\alpha_A) - mgr\cos\alpha_A \\ &= mg\Delta(r\cos\alpha_A) \\ &= -mg\Delta r\cos\alpha_A + mgr\sin\alpha_A\Delta\alpha_A\end{aligned}$$

for small angles

$$\Delta E_p \approx -mg\Delta r\left(1-\frac{\alpha_A^2}{2}\right) + mgr\alpha_A\Delta\alpha_A \quad (3)$$



From conservation of energy

$$\Delta W = \Delta E_p$$

Using equation (2) and (3)

$$\frac{3}{4}mg\alpha_A^2\Delta r + mgr\alpha_A\Delta\alpha_A = 0$$

Rearranging

$$\frac{3}{4}\int\frac{\Delta r}{r} + \int\frac{\Delta\alpha_A}{\alpha_A} = 0$$

$$\frac{3}{4}\ln r + \ln\alpha = const$$

$$r^{3/4}\alpha_A = const$$

Using definition of amplitude $A = \alpha_A r$ results in following adiabatic invariant

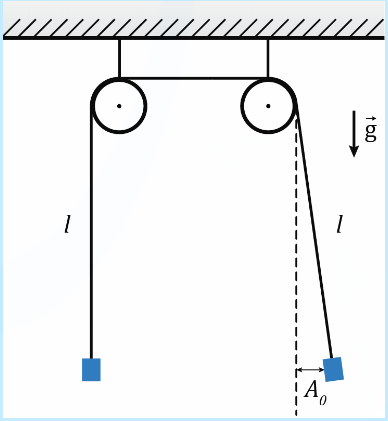
$$\frac{A^4}{r} = const$$

Then amplitude of the small oscillations after reduction of the length of the pendulum by half is

$$\boxed{A_f = \frac{A_0}{\sqrt[4]{2}}}$$

Problem 6

Two weights with identical masses are attached to the weightless string which goes through two pulleys with negligible masses. At initial moment of time a body from the right is pulled sideways by a small distance A_0, which is much less than length of the string l. Find velocity of the left body v at the moment when it will rise up by a distance $l/2$



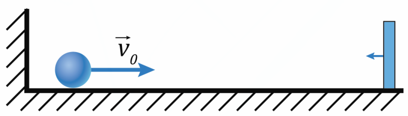
Problem 7

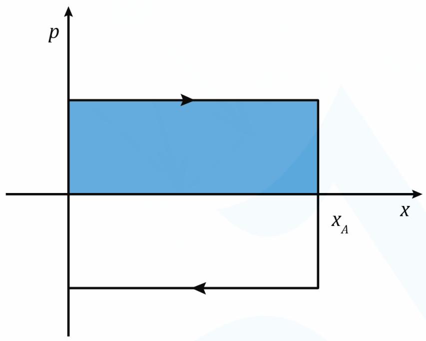
A circuit with a capacitor and inductance coil has initial energy of the system E_0. Parallel plates of the capacitor are very slowly moved apart . What will be energy in the system E_f when distance between plates of the capacitor will double?

Alternative, more straightforward derivations for adiabatic invariants of the mechanical systems is using the fact that

$$I = \oint p(x)dx = const$$

where $p(x)$ is momentum of the system as a function of coordinate x. Contour integral is usually can be replaced with a linear one by leveraging symmetry of the phase diagram (plot of momentum vs coordinate). For example, for a ball bouncing between stationary and slowly moving heavy walls, velocity of the ball v as a function of distance between the walls x_A can be found by examining its phase diagram





From symmetry of the back and forth movements, contour integral is equivalent to

$$I = \oint p(x)dx = 2\int_0^{x_A} mvdx$$

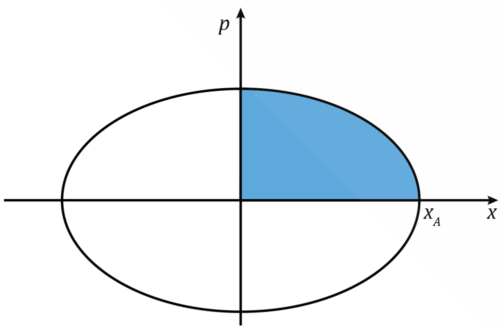
as there is no external forces applied at the ball, its velocity does not dependent of coordinate x). Then

$$I = 2mv\int_0^{x_A} dx = 2mvx_A$$

or was obtained the same result, with one derived earlier by using incremental velocity during collision with a moving wall

$$\boxed{vx_A = const}$$

For cyclic oscillatory systems, but with slowly changing parameters, momentum can be a function of coordinate as



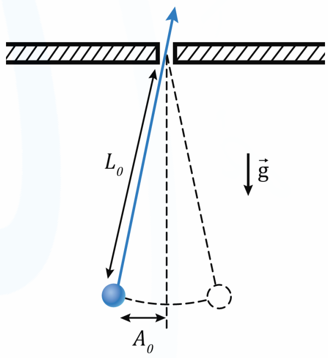
Then contour integral can also be evaluated from symmetry of the phase diagram as

$$I = \oint p(x)dx = 4\int_0^{x_A} p(x)dx = const$$

where x_A is amplitude of oscillations during one cycle

Example 3

A simple pendulum is oscillating in gravity field, while its length is decreasing very slowly by pulling the rope through the hole in the ceiling. What will be amplitude of the small oscillations A_f when the length of the thread will reduce by a half? Initial amplitude of the bob is A_0



Alternative solution to this problem, which is a more straightforward and shorter is by using definition of adiabatic invariant for mechanical system:

$$I = \oint p(x)dx = const \qquad (1)$$

As parameters of the system are changing very slowly, then it can be assumed that energy during one full period of oscillations is conserved. If measuring potential energy of the bob in gravity field from the ceiling, then

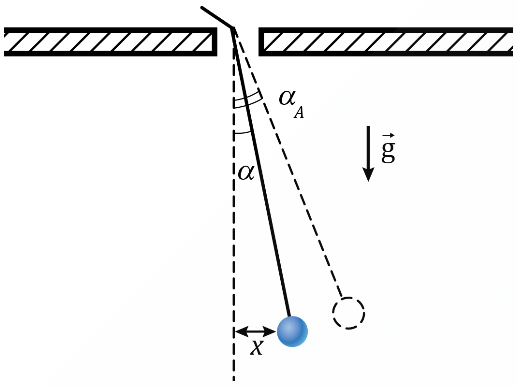
$$-mgr\cos\alpha_A = \frac{p^2}{2m} - mgr\cos\alpha$$

Thus, momentum of the bob as a function of coordinate is

$$p(\alpha) = m\sqrt{gr}\sqrt{\alpha_A^2 - \alpha^2} \qquad (2)$$

where α_A is angular amplitude of the oscillations at that particular period of time when length of the rope is r. For small oscillations $\alpha \ll 1$

$$\sin\alpha \approx \alpha = \frac{x}{r}$$



Using symmetry of the phase diagram, contour integral (1) can be calculated as

$$I = 4\int_0^{\alpha_A} m\sqrt{gr}\sqrt{\alpha_A^2-\alpha^2}\cdot r d\alpha$$

$$= 4m\sqrt{g}\alpha_A^2 r\sqrt{r}\int_0^1\sqrt{1-\frac{\alpha^2}{\alpha_A^2}}d\left(\frac{\alpha}{\alpha_A}\right)$$

If $\alpha/\alpha_A = z$, then

$$\int_0^1\sqrt{1-z^2}dz = C = const$$

so there is no need to evaluate actual integral, no matter how complicated it is, the only answer required is that integral will result in some constant number C. Thus

$$I = 4m\sqrt{g}C\alpha_A^2 r^{3/2} = const$$

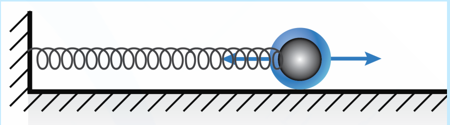
or it is the same result as one obtained by averaging energy changes during one cycle

$$r^3\alpha_A^4 = const$$

$$\boxed{A_f = \frac{A_0}{\sqrt[4]{2}}}$$

Problem 8

A small metallic ball is attached to the spring and the system can oscillate along a flat frictionless horizontal surface with initial amplitude of oscillations A_0. Metallic ball is coated with a layer of ice, which is melting very slowly. What will be amplitude of harmonic oscillations of the bob A_f when total mass of the bob (mass of metallic ball and ice) will reduce by a half?



Problem 9

A small charged ball moves along a flat frictionless surface between two insulating walls A and B. The wall A is fixed, while heavy wall B is moving very slowly to the wall A. Initial distance between walls is L, while kinetic energy of the ball near A is W_A. There is an external constant electric field E applied between two walls. Assume that all collisions are perfectly elastic, charge of the ball q is constant. What will be distance between the walls l at the moment when kinetic energy of the ball near stationary barrier will increase in four times?